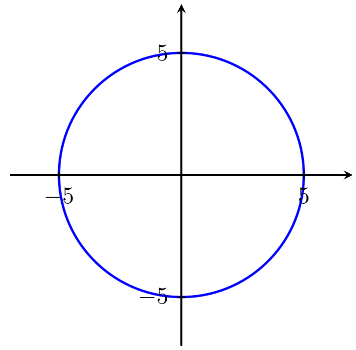
Objectives:

- Find derivatives of implicit functions.

Background:



If we have a formula involving x and y, like

$$x^2 + y^2 = 25,$$

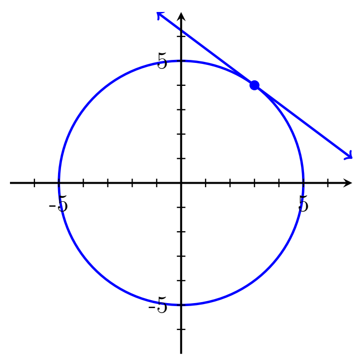
we have a curve that essentially defines y as a function of x near a specific point even though it isn't solved for y. We say y is implicitly a function of x.

Main Idea: We can sneakily find $\frac{dy}{dx}$ (in other words, y') without solving explicitly for y.

How? Differentiate both sides of the equation remembering all the while that y is a function of x.

We will use the chain rule and then solve for y'.

Example 1 Consider $x^2 + y^2 = 25$. Find the slope of the tangent line at the point $(3, 4)$.



Step 1. Differentiate:

$$2x + 2y\frac{dy}{dx} = 0 \quad \text{or} \quad 2x + 2y(y') = 0$$

Step 2. Solve for y':

$$2y(y') = -2x$$
$$y' = \frac{-2x}{2y} = -\frac{x}{y}$$

Step 3. Substitute values: at $(3, 4)$, $\quad y' = -\frac{3}{4}$

Further questions:

1. What is the equation of the tangent line at the point $(3, 4)$?

$$\text{slope: } -\frac{3}{4}, \text{ point: } (3,4), \text{tangent line: } y - 4 = -\frac{3}{4}(x - 3)$$

2. Where is the tangent line horizontal? Vertical?

Horizontal:
Need $y' = -\frac{x}{y} = 0$; $-x = 0$; $x = 0$;
Points: $(0, 5), (0, -5)$

Vertical:
Need y' undefined. Denominator of y' can be zero, so $y = 0$ gives vertical tangent lines.
Points: $(5, 0), (-5, 0)$

Example 2 Find a formula for y' and find where the line tangent to the curve is vertical for the curve given by

$$x^2 + xy + x + y = 1$$

Differentiate:

$$2x + y + y'x + 1 + y' = 0$$

Solve for y':

$$\begin{aligned} y'x + y' &= -2x - y - 1 \\ y'(x+1) &= -2x - y - 1 \\ y' &= \frac{-2x - y - 1}{x+1} \end{aligned}$$

The derivative is undefined when $x = -1$ so if the curve is defined at $x = -1$, the tangent line will be vertical.

Check if the curve is defined at $x = -1$:

$$(-1)^2 - y - 1 + y = 1 - 1 = 0 \neq 1,$$

which is does not satisfy the original equation. So the curve is not defined at $x = -1$.

Since $x = -1$ is the only place the derivative is undefined, there is no point on the curve where the tangent line is vertical.

Example 3 Find the equation of the tangent line to the curve given below at the point $(1, 2)$.

$$x^3 + y^3 + x^2y^2 = 13$$

Check that $(1, 2)$ lies on the curve:

$$1^3 + 2^3 + 1^2 2^2 = 1 + 8 + 4 = 13$$

so $(1, 2)$ does lie on the curve! Differentiate:

$$\begin{aligned} 3x^2 + 3y^2y' + 2xy^2 + x^2 2yy' = 0 \\ y'(3y^2 + 2x^2y) &= -3x^2 - 2xy^2 \\ y' &= \frac{-3x^2 - 2xy^2}{3y^2 + 2x^2y} \end{aligned}$$

At the point $(1, 2)$:

$$y' = \frac{-3 - 2 \cdot 4}{12 + 4} = -\frac{11}{16}$$

So the tangent line is given by

$$y - 2 = -\frac{11}{16}(x - 1)$$